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A Variational Derivation of the Velocity Distribution Functions for Nonequilibrium, Multispecies, Weakly Interacting, Spherically Symmetric Many-Body Systems

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The most probable velocity distribution function of each component, f_a, of a nonequilibrium multispecies spherically symmetric system of particles (stellar plasma atmospheres and winds, stellar systems, pellet-fusion systems) is analytically derived for *the case in which each component is described by the first six moments of* f_a. This is achieved by the aid of a variational approach based on the requirement that the Boltzmann H function for the system be a minimum, subject to the constraints provided by the sets of six macroscopic parameters describing the nonequilibrium state. The use of the so-obtained velocity distribution functions for the closure of the moment equations as well as for the calculation of their collisional terms (via the Fokker–Planck equation) is discussed. The limitations on the maximum deviations from the equilibrium state which are consistent with the assumptions used are also indicated.

KEY WORDS: Velocity distribution functions; plasmas–nonequilibrium; stellar systems–nonequilibrium; weakly interacting many-body systems.

1. INTRODUCTION

The knowledge of the velocity distribution functions of the various components constituting a physical system which is not in equilibrium is of great importance in both laboratory physics and astrophysics. This is particularly true in the case of particles obeying an inverse-square law of interactions (long range potentials), e.g., plasmas (solar corona, solar and stellar winds,

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pellet-thermonuclear fusion) and stellar systems. It is only under very special conditions that such systems can be described by Maxwellian velocity distribution functions.

One of the important uses of the velocity distribution functions is in the derivation of a closed set of macroscopic (fluid) equations which is required for the study of time-dependent nonhomogeneous systems. In such a case, they play a twofold role, namely: (i) they provide relationships between higher and lower moments of the distribution function and consequently close the otherwise infinite chain of equations; and (ii) they enable the calculation of analytical expressions for the terms representing the particle–particle interactions in the moment equations.

In the familiar Chapman–Enskog *kinetic* scheme (see, e.g., Ref. 1), one represents the distribution function as a sum, $f_a = f_a^0 = f_a^{(1)} + f_a^{(2)} \ldots,$ where f_a^0 is the Maxwellian and $f_a^{(i)}$ $(i = 1, 2, \ldots)$ are small perturbations $(|f_a^{(2)}| \ll |f_a^{(1)}| \ll f_a^0)$. Then, one neglects $f_a^{(2)}$ and the other smaller terms and solves an integrodifferential equation for $f_a^{(1)}$. Using this solution (together with the Maxwellian) one obtains a set of equations for the particle density n_a, streaming velocity, $\langle \mathbf{v} \rangle_a$ and random kinetic energy, E_a; the heat flux vector component $\mathbf{q}_a$, appearing in the energy equation is approximated by the value corresponding to the case in which all terms except the one proportional to the temperature gradient in the equation for $\mathbf{q}_a$ are neglected. Eventually, a contribution due to nonzero relative stream velocities is also retained. Thus, only n_a, $\langle \mathbf{v} \rangle_a$, and E_a are treated on an equal footing.

To vindicate this situation, a more general solution for f_a than indicated above is required. For physical systems possessing spherical symmetry in the configuration space and azimuthal symmetry (about the radial direction) in velocity space, it is possible to construct such an approximate solution as follows: Assume that f_a is gyrotropic and can be adequately represented by a sum of zero-order Maxwellian distribution and the first three terms of an expansion in Legendre polynomials (e.g., Ref. 2): $f_a(V, \mu) = f_a^0(V) + \sum_{n=0}^{2} a_n^a(V) P_n(\mu)$ where $P_n(\mu)$ and $a_n^a(V)$ are Legendre polynomials and expansion coefficients (viz., polynomials), respectively. Then, (i) approximating the functions a_n^a as a product of $f_a^0(v)$ by power series in V; (ii) retaining the first three terms for a_0^a, the first two terms for a_1^a and the first term for a_2^a in the corresponding power series; and (iii) imposing the conditions that n_a, $\langle \mathbf{v} \rangle_a$ and E_a be not affected by the non-Maxwellian structure of f_a, one can determine the coefficients in the expansions for a_i^a $(i = 0, 1, 2)$ and consequently the desired f_a's (see e.g., Refs. 3–5). These generalized velocity distribution functions have been heavily used by the last authors for the derivation (for each component, a) of a higher-order, closed set of six fluid equations for the moments $n_a \cdots \xi_{a,r}$ where $\xi_{a,r}$

$\equiv \langle (v_r - \langle v_r \rangle)^4 \rangle - 3[\langle (v_r - \langle v_r \rangle)^2 \rangle]^2$ reflects the effect of a non-Maxwellian tail (excess or deficiency of particles as compared to a Maxwellian).

Now, even though the derivation of the generalized expressions for f_a as sketched above appears plausible and physically meaningful, a more rigorous *kinetic* derivation (i.e., a generalization of the Chapman–Enskog scheme) or a *statistical* derivation (i.e., a generalization of the variational approach used for the derivation of the equilibrium velocity distribution function) is highly desirable.

Thus, in this paper, we use a more general, *statistical* approach for the derivation of nonequilibrium velocity distribution functions in spherically symmetric systems obeying an inverse-square law of interactions such as solar corona, stellar winds, large stellar systems and pellet-thermonuclear fusion. Thus, under circumstances in which relatively high temperatures and low densities prevail, Coulomb collisions are not efficient enough in order to produce thermodynamical or even local thermodynamical equilibrium states. Because of the mathematical complexity of the problem, we are concerned with relatively small deviations from LTE (local Maxwellians). Thus, assuming that each component is adequately represented by its first six moments, n_a, $\langle \mathbf{v} \rangle_a$, $E_a \ldots \xi_{a,r}$, we look for the most probable velocity distribution functions f_a of the form $f_a = f_a(n_a, \langle \mathbf{v} \rangle_a, E_a \ldots \xi_{a,r}; \mathbf{v}; r; t)$. To achieve this, we use a variational approach based on the requirement that Boltzmann's H function for the system be a minimum, subject to the constraints provided by the sets of six macroscopic parameters describing the nonequilibrium state. This procedure represents a generalization of the familiar statistical approach in which the Maxwellian is found to be the most probable velocity distribution function of each component in a system for which the H function is a minimum subject to only three constraints, namely the first three moments (n_a, $\langle \mathbf{v} \rangle_a$, and E_a). (See, e.g., Ref. 6.) The potential extension of this approach to the case of nonequilibrium physical systems along the path sketched above is mentioned, for example, in Refs. 7 and 8.

2. CALCULATIONS

Consider a nonequilibrium many-body system (plasma, star cluster, etc.) which is *spherically symmetric in configuration space* and consists of a number of components (species) each of which is described by the following specific macroscopic quantities (viz., moments of f_a):

number density

$$n_a(r,t) = \int f_a(\mathbf{v}, r, t)\, d^3v \tag{1}$$

radial mean velocity

$$\langle v_r \rangle_a = n_a^{-1} \int v_r f_a(\mathbf{v}, r, t)\, d^3v \tag{2}$$

random kinetic (i.e., thermal) velocity

$$\langle (\mathbf{v} - \langle v_r \rangle_a)^2 \rangle = n_a^{-1} \int (\mathbf{v} - \langle \mathbf{v}_r \rangle)^2 f_a\, d^3v \tag{3}$$

radial flux of random energy

$$q_{r,a}(r,t) = (m_a/2) \int (\mathbf{v} - \langle \mathbf{v}_r \rangle)^2 (v_r - \langle v_r \rangle) f_a\, d^3v \tag{4}$$

the fifth moment of f_a

$$\xi_a(r,t) = n_a^{-1} \int (\mathbf{v} - \langle \mathbf{v}_r \rangle)^4 f_a\, d^3v \tag{5}$$

thermal anisotropy

$$A_a(r,t) = n_a^{-1} \int \left[(v_r - \langle v_r \rangle)^2 - v_\perp^2 \right] f_a\, d^3v \equiv \alpha_a - \beta_a \tag{6}$$

In the above six equations we used spherical polar coordinates, r, θ, and ϕ as well as the notation $v_\theta = v_\phi \equiv v_\perp$ (because of the spherical symmetry).

In the continuation, for convenience, we use the following new variables:

$$\begin{aligned} &\mathbf{V} \equiv \mathbf{v} - \langle \mathbf{v}_r \rangle_a \\ &V_r \equiv (\mathbf{v} - \langle \mathbf{v}_r \rangle_a)_r = V\mu, \qquad V_\perp = \left[(1 - \mu^2)/2 \right]^{1/2} V \end{aligned} \tag{7}$$

Here μ is the cosine of the angle between the vector $\mathbf{V}$ and the radial direction. With these variables, Eqs. (1)–(6) read (for simplicity we omit the subscript a, indicating the specific plasma component):

$$n = \int f\, d^3V \tag{1'}$$

$$\langle v_r \rangle = n^{-1} \int V\mu f\, d^3V \tag{2'}$$

$$\langle \mathbf{V}^2 \rangle = n^{-1} \int V^2 f\, d^3V \tag{3'}$$

$$q_r = 0.5m \int V^3 \mu f\, d^3V \tag{4'}$$

$$\xi = n^{-1} \int V^4 f\, d^3V \tag{5'}$$

and

$$A = n^{-1} \int V^2 P_2(\mu) f\, d^3V \tag{6'}$$

where $P_2(\mu) \equiv 0.5(3\mu^2 - 1)$ is the Legendre polynomial of order two, $f = f(V, \mu, r, t)$ and $d^3V = -V^2 d\mu\, d\phi\, dV$.

Now, we want to find the most probable velocity distribution function of each component, f_a, subject to the constraints provided by the six macroscopic parameters describing the nonequilibrium state, as indicated above. (We anticipate that the sought-for nonequilibrium distribution functions will not have Maxwellian shapes and will not satisfy the equilibrium conditions $f_a'f_b' = f_a f_b$ for all a and b.)

To solve the problem posed above, we shall use a rather general variational approach by requiring that the Boltzmann H function be a minimum under our six restricting conditions.

First, from the definitions

$$H = \sum_a H_a, \qquad H_a \equiv \int f_a \ln f_a\, d^3V \tag{8}$$

it is seen that if we can minimize each H_a we shall also minimize H.

Second, using for the moment a discrete representation for f_a, one can write the following expressions for H_a and the six quantities defined by Eqs. (1′)–(6′):

$$H = \sum_i f_i \ln f_i \tag{9}$$

$$n = \sum_i f_i \tag{10}$$

$$\langle V_r \rangle = n^{-1} \sum_i V_i \mu_i f_i \tag{11}$$

$$\langle \mathbf{V}^2 \rangle = n^{-1} \sum_i V_i^2 f_i \tag{12}$$

$$q_r = 0.5 m \sum_i V_i^3 \mu_i f_i \tag{13}$$

$$\xi = n^{-1} \sum_i V_i^4 f_i \tag{14}$$

$$A = n^{-1} \sum_i V_i^2 P_2(\mu_i) f_i \tag{15}$$

Using now the method of the Lagrange multipliers we can write for each variable f_i the following equation:

$$\frac{\partial}{\partial f_i} \left\{ \sum_k f_k \ln f_k - \left[\lambda_1 f_k + \lambda_2 V_k \mu_k f_k + \lambda_3 V_k^2 f_k + \lambda_4 V^4 f_k + \lambda_5 V_k^3 \mu_k f_k + \lambda_6 V_k^2 P_2(\mu_k) f_k \right] \right\} = 0 \tag{16}$$

[The constants in the equations (10)–(15) have been included in the λ's.]

Solving Eq. (16) for f_i and going back to the continuous function, we obtain

$$f(V, \mu) = \exp\left[\bar{\lambda}_1 + \lambda_2 V\mu + \lambda_3 V^2 + \lambda_4 V^4 + \lambda_5 V^3\mu + \lambda_6 V^2 P_2(\mu)\right] \quad (17)$$

where $\bar{\lambda}_1 = \lambda_1 - 1$.

For convenience we write the constants λ_1, λ_2, and λ_3 as follows:

$$\bar{\lambda}_1 = \lambda_1' + \lambda_1'', \qquad \lambda_2 = \lambda_2' + \lambda_2'', \qquad \lambda_3 = -\lambda_3' + \lambda_3'' \quad (18)$$

Using this decomposition in Eq. (17) and assuming that λ_1'', $\lambda_2'' V\mu$, $\lambda_3'' V^2$, $\lambda_4 V^4$, $\lambda_5 V^3\mu$, and $\lambda_6 V^2 P_2(\mu)$ are small compared to one, one can expand the expression (17) in a Taylor series and bring it to the following form:

$$f(V, \mu) = \left[1 + \lambda_1'' + \lambda_2'' V\mu + \lambda_3'' V^2 + \lambda_4 V^4 + \lambda_5 V^3\mu + \lambda_6 V^2 P_2(\mu)\right] f^0 \quad (19)$$

where

$$f^0 = \exp(\lambda_1' + \lambda_2' V\mu - \lambda_3' V^2) \quad (20)$$

Recalling that the first two Legendre polynomials are $P_0 = 1$ and $P_1 = \mu$ one may write Eq. (19) as a sum of a "Maxwellian-type" velocity distribution function and three other terms, as follows:

$$f = f^0 + (\lambda_1'' + \lambda_3'' V^2 + \lambda_4 V^4) f^0 P_0 + \left[(\lambda_2'' + \lambda_5 V^2) V\right] P_1 f^0 + \lambda_6 V^2 P_2 f^0 \quad (21)$$

Alternatively, we can write Eq. (21) as

$$f = f^0 + \lambda_1''\left[1 + (\lambda_3''/\lambda_1'') V^2 + (\lambda_4/\lambda_1'') V^4\right] P_0 f^0 + \lambda_2''\left[1 + (\lambda_5/\lambda_2'') V^2\right] V P_1 f^0 + \lambda_6 V^2 P_2 f^0 \quad (21')$$

Equation (21) contains nine constants, namely, $(\lambda_1', \lambda_2', \lambda_3')$, $(\lambda_1'', \lambda_2'', \lambda_3'')$, and $(\lambda_4, \lambda_5, \lambda_6)$. They can be determined by the aid of the *six* constraints represented by Eqs. (1)–(6) and of three additional requirements which we impose on the perturbation $f' \equiv f - f^0$, namely,

$$\int f' \, d^3V = 0 \quad (22)$$

$$\int f' V_r \, d^3V = 0 \quad (23)$$

$$\int f' V^2 \, d^3V = 0 \quad (24)$$

The conditions (22)–(24) simply state that the nonequilibrium deviations from the Maxwellian do not affect the particle density, radial streaming velocity, and kinetic energy of random motion.

Thus, proceeding as indicated above we obtain the following values for the nine constants λ:

$$\lambda_1' = \ln(\lambda_3'/\pi)^{3/2}$$

$$\lambda_2' = 0 \qquad (25)$$

$$\lambda_3' = 3/2\langle V^2\rangle$$

$$\lambda_3'' = -(2/3B)\lambda_1''$$

$$\lambda_4 = (1/15B^2)\lambda_1'' \qquad (26)$$

$$\lambda_5 = -(1/5B)\lambda_2''$$

$$\lambda_1'' = (\xi - 15B^2)/8B^2 \qquad (27)$$

$$\lambda_2'' = -q_r/nmB^2 \qquad (28)$$

$$\lambda_6 = A/3B = (\alpha - \beta)/3B \qquad (29)$$

Notice that by (25) Eq. (20) reads

$$f^0 = (n/2\pi B)^{3/2}\exp(-V^2/2B) \qquad (30)$$

where $mB \equiv m\langle v^2\rangle/3$ is the average kinetic energy of random motion ($mB \equiv E/3$).

Next, we define two quantities which are related to the moments $q_{a,r}$ and ξ_a as follows:

$$\epsilon_{a,r}(r,t) = \langle(v_r - \langle v_r\rangle_a)^3\rangle_a \qquad (31)$$

and

$$\xi_{a,r}(r,t) = \langle(v_r - \langle v_r\rangle_a)^4\rangle_a \qquad (32)$$

The relations between $\epsilon_{a,r}$, $\xi_{a,r}$, and $q_{a,r}$, ξ_a are as follows:

$$q_{a,r} = (5m_a n_a/6)\epsilon_{a,r} \qquad (33)$$

and

$$\xi_a = 5[\xi_{a,r} - 4(\alpha_a - \beta_a)B_a] \qquad (34)$$

where we used $A_a \equiv \alpha_a - \beta_a$. [See Eq. (6).] For convenience, we shall use a "modified" fifth moment, namely,

$$\xi_{a,r}(r,t) \equiv \langle(v_r - \langle v_r\rangle_a)^4\rangle - \langle(v_r - \langle v_r\rangle_a)^4\rangle_{a,\mathrm{Maxw}}$$
$$= \xi_{a,r}(r,t) - 3\alpha_a^2(r,t) \qquad (35)$$

Here, $3\alpha^2$ is the value of $\xi_{a,r}$ in the case of a Maxwellian [$\alpha \equiv \langle(v_r - \langle v_r\rangle_a)^2\rangle_a$ is the radial random mean square velocity]. By (25)–(29) and (31),

(32) one obtains

$$\lambda''_{a,1} = \frac{5}{8} \frac{\xi_{a,r} + 4(\alpha_a - \beta_a)^{2/3}}{B_a^2} \equiv c_0^a \tag{36}$$

$$\lambda''_{a,2} = -\frac{5}{6} \frac{\epsilon_{a,r}}{B_a^2} \equiv -c_1^a / B^{1/2}, \tag{37}$$

$$\lambda_{a,6} = \frac{1}{3} \frac{\alpha_a - \beta_a}{B_a} \equiv c_2^a / B \tag{38}$$

Notice that the results (36)–(38) completely determine the values of λ''_3, λ_4 and λ_5 [Eqs. (26)].

Now, substituting the results (26)–(38) into Eq. (21), after some algebra we obtain the result

$$f_a(n_a, \langle v_r \rangle_a, \alpha_a, \beta_a, \epsilon_{a,r}, \xi_{a,r}; \mathbf{V}, r, t)$$
$$= f^0(n_a, \langle v_r \rangle_a, B_a; \mathbf{V}, r, t) + \sum_{n=0}^{2} a_n^a(V) P_n(\mu) \tag{39}$$

where f^0, the local Maxwellian distribution function, is given by Eq. (30) with $B_a = (\alpha_a + 2\beta_a)/3$ and n_a, $\langle v_r \rangle$, B_a being functions of r and t. The functions $a_i^a(V)$ are given by

$$\begin{aligned} a_0^a(V) &= c_0^a \left(1 - \frac{2}{3} \frac{V^2}{B_a} + \frac{1}{15} \frac{V^4}{B^2}\right) f^0 \\ a_1^a(V) &= c_1^a \left(-1 + \frac{1}{5} \frac{V^2}{B_a}\right) \frac{V}{B_a^{1/2}} f^0 \\ a_2^a(V) &= c_2^a \frac{V^2}{B_a} f^0 \end{aligned} \tag{40}$$

With c_0^a, c_1^a, and c_2^a given by (36)–(38).

Finally in terms of the more familiar fluid quantities $T_r \equiv m\alpha/K$, $T \equiv m\beta/K$, and $q_r \equiv (5/6) mn\epsilon_r$, f reads

$$\begin{aligned} f = f_{\text{Maxwell}} \Big\{ 1 &+ \tfrac{5}{8} \left[\bar{\xi}_r + 4(T_r - T_\perp)^2 / 3T \right] \left(1 - \tfrac{2}{3} \bar{V}^2 + \tfrac{1}{15} \bar{V}^4\right) \\ &+ \bar{q}_r \bar{V}\left(-1 + \tfrac{1}{5} \bar{V}^2\right)\mu + \left[(T_r - T_\perp)/3T\right] \bar{V}^2 \left(\tfrac{3}{2}\mu^2 - \tfrac{1}{2}\right) \Big\} \end{aligned} \tag{41}$$

where

$$\bar{V} = \frac{V}{(KT/m)^{1/2}}, \qquad T = \frac{(T_r + 2T_\perp)}{3}$$

$$\bar{\xi}_r = \frac{\xi_r}{(KT/m)^2}, \qquad \bar{q}_r = \frac{q_r}{nm(KT/m)^{3/2}}$$

We recall that to obtain these results we assumed relatively small deviations from the local equilibrium state. This assumption was introduced through the expansion leading to Eq. (21) which implies that the quantities λ_1'', $\lambda_2'' V\mu$, $\lambda_3'' V^2$, $\lambda_4 V^4$, $\lambda_5 V^3 \mu$, and $\lambda_6 P_2(\mu) V^2$ are small compared to unity. After expressing the λ's in terms of the moments of f and obtaining the explicit expression (41) for f, we see that the consistency condition requires the smallness (compared to one) of the terms in the curled brackets of (41). Thus, the results hold for relatively small thermal conduction,[3] q_r (or equivalently small skewness of f), thermal anisotropy, $T_r - T_\perp$ and deviation from the Maxwellian tail, ξ.

Finally, while the skewness and thermal anisotropy of f do not require supplementary explanations, the effect on f due to ξ_r needs some special attention. Thus, consider only the term proportional to ξ_r in (41). Then, as is easily seen, the expression multiplying $\bar{\xi}_r$, say Q, has two roots, namely, $\bar{v}_1 \simeq 1.36$ and $\bar{v}_2 \simeq 2.86$. Thus $Q > 0$ in the ranges (A) $0 < \bar{v} < 1.36$ and (C), $\bar{v} > 2.86$; also, $Q < 0$ in the range (B), $1.36 < \bar{v} < 2.86$. Moreover, the evaluation of ξ_r $[\xi_r \equiv n^{-1} \int (v_r - \langle v_r \rangle)^4 (f - f_{\text{Maxwell}})\, d^3v \propto \int_0^\infty S(v)\, dv]$ provides $\bar{\xi}_{r,A}/C_0 \simeq 0.021$, $\bar{\xi}_{r,B}/c_0 \simeq -0.911$ and $\bar{\xi}_{r,C}/c_0 \simeq 2.509$. Here $\xi_{r,A} = \int_0^{v_1} S(v)\, dv$, $\xi_{r,B} = \int_{v_1}^{v_2} S(v)\, dv$, $\xi_{r,C} = \int_{v_2}^{\infty} S(v)\, dv$. Thus, practically, ξ does not affect the very low energy particles (bulk particles); however, it implies a depletion of intermediate energy particles (range B) and an enhancement of the tail particles (elongated tail, region C). (We mean changes with respect to a Maxwellian, $\xi_{r,\text{Maxwell}} = 0$). Since $\xi_{r,C} > |\xi_{r,B}|$ and the tail population is significantly lower than the population of regions $A + B$, the effect of $\xi_r \neq 0$ on the tail particles can be rather important.

3. DISCUSSION AND SUMMARY

Using a variational approach based on the minimization of Boltzmann's H function we obtained the most probable distribution function, f_a, for a spherical nonequilibrium system of Coulomb interacting particles (plasmas, stars) defined by the first six velocity moments of f_a. The result can also be expressed as the sum of a Maxwellian and the first three terms of an expansion in Legendre polynomials. The coefficients of the Legendre polynomials are themselves polynomials in the relative velocity, $|\mathbf{v} - \langle \mathbf{v}_r \rangle|$ and are multiplied by the Maxwellian; they are also proportional to quantities involving combinations of the higher-order moments, $q_{a,r}$, $\xi_{a,r}$, and $(T_r - T_\perp)$.

The constrained minimization used in this work (in which f depends

[3] We notice that even in the simple three-moments equations approach in which the Chapman–Enskog procedure is used in order to calculate q_r, one assumes $f_1 \ll f_0$ and therefore the relative smallness of q_r.

on its first six moments) and leading to our result represents a generalization of the familiar one leading to the Maxwellian (in which f depends on its first three moments) which is usually used in the type of problems which motivated this work. Thus, it provides more information about the system, through the additional moments it contains. On the other hand, more refined kinetic aspects connected with the fine structure of f and pertinent to resonant processes are not treatable within the framework of the present theory. For this, a fully kinetic description, or equivalently, the consideration of an infinite number of moments of f would be required. Unfortunately, such an ambitious approach is impractical for the solution of realistic problems, as those which motivated this work.

Finally, the expression for the most probable velocity distribution function obtained here enables one to derive—starting from Boltzmann's equation—a closed set of moments equations as follows. First, it relates higher (than sixth) order moments to lower-order moments, thus cutting the infinite chain of moments equations. Secondly, it enables the calculation—based on Fokker–Planck formalism—of the collisional transport coefficients appearing in the moments equations. Such a closed system of equations is absolutely necessary for the description of physical systems such as stellar-plasma atmospheres and winds, stellar systems, pellet fusion systems, etc.

Actually, we have already derived such equations by using as an ansatz for f a form of the type obtained in the present paper.[4,5] The present work provides a rigorous proof and justification of the ansatz mentioned above.[9–10]

The next step will consist of a numerical solution of the six moments equations (for each species). As a result, the detailed time and space behavior of the sets of six moments of the f's, as well as of the f's themselves, will be, hopefully, obtained.

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